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# The Long-Term Model of Salinity Intrusion into the Estuarine Rivers

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<https://dx.doi.org/doi:10.25773/v5-rktn-h635>

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THE LONG-TERM MODEL OF SALINITY  
INTRUSION, INTO THE ESTUARINE RIVERS

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A THESIS

Presented to

The Faculty of the School of Marine Science  
The College of William and Mary in Virginia

In Partial Fulfillment  
Of the Requirements for the Degree of  
Master of Arts

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by

Mary Ann Terese Orzech


1972 //

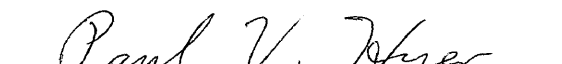
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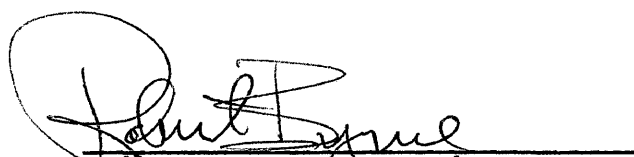
This thesis is submitted in partial fulfillment of  
the requirements for the degree of  
Master of Arts in Marine Science

  
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Approved, September 1972

  
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## ACKNOWLEDGMENTS

The author wishes to express appreciation to her major professor, Dr. Albert Y. Kuo, for his patient guidance and encouragement during this investigation. Thanks and appreciation are also extended to Dr. Michael Bender, Dr. Paul V. Hyer, Dr. Robert Byrne and Dr. William MacIntyre for their criticism of the manuscript, and to Mr. Richard Moncure for providing assistance with the computer programming.

The author also wishes to thank the Department of Physical Oceanography and Hydraulics of the Virginia Institute of Marine Science for the salinity, hydrographic, and dye experiment data.

The major funding of this study was provided by the Division of Water Resources and State Water Control Board of Virginia; their support is greatly appreciated.

Special thanks are extended to Mrs. Shirley Crossley, who patiently typed this manuscript and the endless equations, and to Mrs. Jane Davis, who prepared the figures.

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## ABSTRACT

A rigorous development of the long-term salinity model is presented. This model describes the salinity distribution averaged over a tidal cycle or the salinity distribution at slack water along an estuarine river.

To derive the model, the mass balance equation was averaged over a tidal cycle. The resulting dispersion coefficient consisted of two components due to dispersion by the shear effect and the oscillating tidal currents. Analysis of field data showed that the dispersion due to the oscillating tidal currents was more important than the dispersion due to the shear effect.



THE LONG-TERM MODEL OF SALINITY INTRUSION  
INTO THE ESTUARINE RIVERS

# I

## INTRODUCTION

An estuary has been described as a partially enclosed body of water which has free access to the open sea and where freshwater from the land meets the sea water. The saline layer from the sea intrudes upstream underneath the freshwater layer from the rivers. In a well-mixed tidal estuary, the turbulence induced by the tidal current mixes the fresh and salt water and reduces the density gradients within a cross-section of the estuary. The oscillating tidal current is the primary mechanism responsible for the transport of salt upstream.

Salinity is a distinctive parameter of the estuarine environment which influences the circulation patterns, the density stratification and the distribution of marine organisms along the estuary. Furthermore, the hydrodynamic behavior of an estuary can be investigated by following the distribution of salt, a natural tracer of water movements. The transport of a conservative dissolved substance, such as salt, in turbulent flow may be represented by the mass balance equation

$$\begin{aligned} \frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} + w \frac{\partial c}{\partial z} = \frac{\partial}{\partial x} \left[ e_x \frac{\partial c}{\partial x} \right] + \frac{\partial}{\partial y} \left[ e_y \frac{\partial c}{\partial y} \right] \\ + \frac{\partial}{\partial z} \left[ e_z \frac{\partial c}{\partial z} \right] \end{aligned} \quad (I-1)$$

where  $t$  is time,  $c(x,y,z,t)$  is the mean concentration,  $u$ ,  $v$  and  $w$  are the mean velocity components in the  $x$ ,  $y$  and  $z$  directions respectively, and  $e_x$ ,  $e_y$ , and  $e_z$  are the turbulent diffusion coefficients. The mean quantities are the ensemble mean or time mean over the proper interval.\*

The concentration field as a function of time and three spatial dimensions is difficult to solve either analytically or numerically, even with the improved capabilities of the modern computer. In some practical applications to estuarine rivers, the problems may be simplified by seeking information on the average concentration over the cross section. This is achieved by integrating the mass balance equation over a cross-section of the estuary normal to its axis. Integrating equation (I-1), we have

$$\begin{aligned} \frac{\partial}{\partial t} (AC) + \frac{\partial}{\partial x} (AUC) + \frac{\partial}{\partial x} \int_A u''c''dA \\ = \frac{\partial}{\partial x} \left[ A\bar{e}_x \frac{\partial C}{\partial x} \right] \end{aligned} \quad (I-2)$$

where  $A$  is the cross-sectional area,  $U$  is the longitudinal

---

\*

It is impossible to obtain a true ensemble mean in a turbulent flow occurring in nature. A time average over a period longer than the turbulent time scale while shorter than the time scale of gross variation may serve to approximate an ensemble mean.

velocity averaged over a cross-section,  $C$  is the concentration averaged over a cross-section,  $x$  is the distance along the axis,  $u''$  and  $c''$  are the spatial deviations of the velocity component and the concentration respectively from the cross-sectional average and  $\bar{e}_x$  is the spatial mean value of the turbulent diffusivity.

The integral  $\int_A u''c''dA$  represents the mass transport associated with the velocity variation over the cross-section or shear effect. Taylor (1953, 1954) and Aris (1956) showed that this mass transport may be described approximately by Fickian type diffusion and so it is possible to write

$$\int_A u''c''dA = -AE \frac{\partial C}{\partial x} \quad (I-3)$$

where  $E$  is the dispersion coefficient.

After substituting equation (I-3) into equation (I-2), equation (I-2) may be written as

$$\frac{\partial}{\partial t} (AC) + \frac{\partial}{\partial x} (AUC) = \frac{\partial}{\partial x} \left[ A(E + \bar{e}_x) \frac{\partial C}{\partial x} \right] \quad (I-4)$$

where  $E$  is usually much larger than  $\bar{e}_x$ . Harleman (1971) defined the sum of  $E$  and  $\bar{e}_x$  as the longitudinal dispersion coefficient  $E_D$ ,

$$E_D = E + \bar{e}_x \quad (I-5)$$

In general, the cross-sectional average of the concentration  $C$ , of the velocity  $U$ , and the area are functions of

x and t. In an estuarine river, the velocity U is due to freshwater discharge and the tidal current.

Taylor (1953), who examined laminar flow in a circular tube, was the first to determine analytically a value for the dispersion coefficient E. He demonstrated, both analytically and experimentally, that the average concentration of a dissolved substance over the cross-section of the tube was dispersed relative to a plane moving with the mean velocity U as though it obeyed a Fickian type diffusion equation. The dispersion coefficient was found to be

$$E = \frac{a^2 U^2}{48 D}$$

where D is the coefficient of molecular diffusion, a is the radius of the tube and U is the mean velocity over a cross-section of the tube. The combined action of molecular diffusion in the radial direction and the variation in velocity over the cross-section of the tube, created by friction at the walls, produced this dispersion.

Taylor (1954), in an analogous treatment, extended this theory to turbulent flow in a pipe. The effective longitudinal dispersion coefficient was found to be  $10.1 au_*$  where  $u_*$  is the shear velocity given by  $u_* = (\tau_o / \rho)^{1/2}$ ,  $\tau_o$  being the shear stress at the wall and  $\rho$ , the density of the fluid.

Elder (1959) extended Taylor's theory to steady, uniform flow in a wide open channel. The longitudinal dispersion coefficient was computed to be  $5.9 u_* h$  where  $h$  was the depth of the fluid.

Bowden (1965) evaluated the dispersion coefficients for various velocity profiles in a channel of uniform depth. He derived the expression

$$E = khU$$

with the constant  $k$  dependent on the velocity profile.

The dispersion characteristics in natural streams and their dependence on the bulk parameters of the channel was studied by Fischer (1967). He demonstrated that the lateral velocity variation rather than vertical shear was the primary dispersive mechanism in natural streams with large width-to-depth ratios.

Okubo (1967) compared the shear effect in oscillatory flow to that of steady flow, the magnitude of which equaled the amplitude of the oscillating current. He concluded that if the length of time required to produce vertical homogeneity by mixing in a bounded sea was much longer than the period of oscillation, the shear effect of the oscillating current was negligible compared to the steady current. If the vertical mixing was accomplished within the period of oscillation, the shear effect produced by the periodic motion became as important as that for steady flow.

Holley, et. al. (1970) studied dispersion in one-dimensional oscillatory flow and the dependence of dispersion on the parameter  $T'$ , the ratio between the period of oscillation  $T$  and the time scale for cross-sectional mixing  $T_C$ . If  $T' \gg 1$ , the quasi-steady approach may be used and the average dispersion coefficient over a tidal cycle,  $\bar{E}$ , can be related to the average hydraulic conditions during the period of oscillation. If  $T' < 1$ ,  $E$  varied approximately as the square of  $T'$ . The dispersions due to vertical shear and transverse shear were treated independently. The parameter  $T_V'$  was evaluated based on  $T_C = T_V$ , the time scale of vertical mixing, for the vertical shear effect, and  $T_t'$  was evaluated based on  $T_C = T_t$ , the time scale of transverse mixing, for the transverse shear effect. They observed that for most estuaries  $T_V'$  was usually greater than unity but  $T_t'$  was much less than unity. As the width of the channel increased, the dispersion due to transverse shear decreased with the square of  $T_t'$ . Thus they concluded that vertical shear was the dominant dispersive mechanism in wide estuaries and that the dispersion coefficient could be approximated by the expression

$$\bar{E} = bhu_* \quad (I-6)$$

where  $u_*$  was the shear velocity related to the tidal velocity averaged over one-half period, 'b' was a constant and  $h$  was the uniform depth.

The investigations of Taylor (1954), Elder (1959),

Bowden (1965) and Holley, et. al. (1970) suggest that the dispersion coefficient may be represented by the expression

$$E = k h |U| \quad (I-7)$$

With the dispersion coefficient expressed in terms of  $U$  and  $h$ , equation (I-4) may be applied to an estuarine river and solved numerically with a digital computer. In the formulation of numerical computations with finite difference approximation, two distinct types of models arise due to the oscillating feature of the convective current  $U$ : the short-term model or real-time model and the long-term or slack tide approximation model. The purpose of analysis and the response time of the system are important criteria in determining the type of model.

In a real-time model the time step of numerical computation is much smaller than a tidal period, thus the time variation of the tidal component of the convective velocity may be included in the model. The real-time model is used when the response time of the system is short and an equilibrium state is reached quickly. This model is also used when short-term variations of a concentration field are to be investigated.

The time step is an integral multiple of the tidal period for a long-term model. In this model the convective velocity is the velocity averaged over a tidal period, i.e. the non-tidal component. The convection due to the tidal



current is incorporated into the dispersion term. The dispersion coefficient should include the contribution from the transport by the oscillating tidal current in addition to the value given by averaging equation (I-7) over a tidal cycle. Since the computation time required would be less, this model is more practical for the investigation of long-term variations of a concentration field.

In this investigation, a rigorous development of the longitudinal dispersion coefficient for a long-term salinity model is presented. This model describes the salinity distribution averaged over a tidal cycle or the salinity distribution at slack water along an estuarine river.

## II

### MATHEMATICAL FORMULATION

Applying equation (I-4) to the transport of saline water in an estuarine river and neglecting the turbulent diffusion with respect to dispersion, the one-dimensional mass balance equation becomes

$$\frac{\partial}{\partial t} (AS) + \frac{\partial}{\partial x} (AUS) = \frac{\partial}{\partial x} \left( AE \frac{\partial S}{\partial x} \right) \quad (\text{II-1})$$

where  $S$  is the salt concentration averaged over a cross-section. The dispersion coefficient  $E$  may be expressed as

$$E = kh|U| \quad (\text{II-2})$$

for most estuarine rivers with large width-to-depth ratios. Equation (II-1) may be used to construct a finite difference scheme for the real-time model of salinity intrusion.

Because of the periodicity of the velocity  $U$ , equation (II-1) cannot be directly approximated by a finite difference form with a time increment larger than a tidal cycle. In a long-term model, the time increment of the numerical computations is an integral multiple of the tidal cycle and the parameters in each step of computation take representative values during that time interval. Equation (II-1) has to be averaged over a tidal cycle before a long-term model may be constructed.

The cross-sectional averages of the salt concentration

S, the velocity U, and the area A can be written as the sum of a mean and a temporal deviation. The relationship is expressed as follows:

$$\begin{aligned} S &= \bar{S} + S' \\ U &= \bar{U} + U' \\ A &= \bar{A} + A' \end{aligned} \quad (\text{II-3})$$

where the overbars represent the mean over a tidal cycle and the primes represent the temporal deviations. So, by substituting equations (II-3) into equation (II-1) and averaging it over a tidal cycle, we have

$$\begin{aligned} & \frac{\partial}{\partial t} \left[ \overline{\bar{A}S} + \overline{\bar{A}S'} + \overline{A'\bar{S}} + \overline{A'S'} \right] \\ & + \frac{\partial}{\partial x} \left[ \overline{\bar{A}U\bar{S}} + \overline{\bar{A}U\bar{S}'} + \overline{\bar{A}U'\bar{S}} + \overline{\bar{A}U'S'} + \overline{A'\bar{U}\bar{S}} + \overline{A'\bar{U}\bar{S}'} + \overline{A'U'\bar{S}} + \overline{A'U'S'} \right] \\ & = \frac{\partial}{\partial x} \left( \overline{A\bar{E} \frac{\partial S}{\partial x}} \right) \end{aligned} \quad (\text{II-4})$$

Okubo (1964) proposed that Reynolds axioms are approximately satisfied by the time average operator over one tidal cycle since the mean values of the velocity  $\bar{U}$ , salt content  $\bar{S}$ , and cross-sectional area  $\bar{A}$  in estuaries can be considered slowly varying functions of time. It then follows from Reynolds axioms that

$$\begin{aligned} \overline{\bar{A}S} &= \bar{A}\bar{S}, \quad \overline{\bar{A}S'} = \overline{\bar{A}S'} = 0, \quad \overline{A'\bar{S}} = \overline{A'\bar{S}} = 0, \\ \overline{\bar{A}U\bar{S}} &= \bar{A}\bar{U}\bar{S}, \quad \overline{\bar{A}U\bar{S}'} = \overline{\bar{A}U\bar{S}'} = 0, \quad \overline{\bar{A}U'\bar{S}} = \overline{\bar{A}U'\bar{S}} = 0, \\ \overline{A'\bar{U}\bar{S}} &= \overline{A'\bar{U}\bar{S}} = 0 \end{aligned}$$

Thus, equation (II-4) reduces to

$$\begin{aligned}
 & \frac{\partial}{\partial t} (\overline{AS} + \overline{A'S'}) \\
 & + \frac{\partial}{\partial x} (\overline{AUS} + \overline{A} \overline{U'S'} + \overline{A'S'} \overline{U} + \overline{A'U'} \overline{S} + \overline{A'U'S'}) \\
 & = \frac{\partial}{\partial x} \left( \overline{AE \frac{\partial S}{\partial x}} \right) \quad (II-5)
 \end{aligned}$$

In most parts of an estuarine river, the salinity and cross-sectional area fluctuations are small compared with their mean values, i.e.  $|S'| \ll \overline{S}$ ,  $|A'| \ll \overline{A}$ , while the velocity fluctuation is usually much larger than its mean. Thus, by neglecting the second and higher order terms of  $S'$  and  $A'$ , equation (II-5) becomes

$$\begin{aligned}
 & \frac{\partial}{\partial t} (\overline{AS}) + \frac{\partial}{\partial x} (\overline{AUS} + \overline{A} \overline{U'S'} + \overline{S} \overline{A'U'}) \\
 & = \frac{\partial}{\partial x} \left( \overline{AE \frac{\partial S}{\partial x}} \right) \quad (II-6)
 \end{aligned}$$

Furthermore, equation (II-6) can be rewritten as

$$\begin{aligned}
 & \frac{\partial}{\partial t} (\overline{AS}) + \frac{\partial}{\partial x} (\overline{AUS} + \overline{S} \overline{A'U'}) \\
 & = \frac{\partial}{\partial x} \left[ \overline{AE \frac{\partial S}{\partial x}} \right] + \frac{\partial}{\partial x} \left[ \overline{A} (-\overline{U'S'}) \right] \quad (II-7)
 \end{aligned}$$

Next, neglect lateral inflow and average the one-dimensional continuity equation

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x} (AU) = 0 \quad (\text{II-8})$$

over a tidal cycle. This gives

$$\frac{\partial \bar{A}}{\partial t} + \frac{\partial}{\partial x} (\bar{A}\bar{U} + \overline{A'U'}) = 0 \quad (\text{II-9})$$

Now, expand equation (II-7) and substitute the tidal averaged continuity equation (II-9) into the result to further simplify equation (II-7). These operations reduce equation (II-7) to

$$\begin{aligned} & \frac{\partial \bar{S}}{\partial t} + \frac{1}{\bar{A}} (\bar{A}\bar{U} + \overline{A'U'}) \frac{\partial \bar{S}}{\partial x} \\ &= \frac{1}{\bar{A}} \frac{\partial}{\partial x} \left[ \overline{AE \frac{\partial S}{\partial x}} \right] + \frac{1}{\bar{A}} \frac{\partial}{\partial x} \left[ \bar{A} (-\overline{U'S'}) \right] \end{aligned} \quad (\text{II-10})$$

Pritchard (1958) derived an expression for the average freshwater discharge over a tidal period  $\bar{Q}$  as

$$\bar{Q} = \overline{AU} = \bar{A}\bar{U} + \overline{A'U'} \quad (\text{II-11})$$

and he defined freshwater velocity  $U_f$  as

$$U_f = \frac{\bar{Q}}{\bar{A}} = \bar{U} + \frac{\overline{A'U'}}{\bar{A}} \quad (\text{II-12})$$

Employing these relationships simplifies the mass balance equation (II-10) to

$$\frac{\partial \bar{S}}{\partial t} + U_f \frac{\partial \bar{S}}{\partial x} = \frac{1}{\bar{A}} \frac{\partial}{\partial x} \left[ \overline{AE \frac{\partial S}{\partial x}} + \bar{A} (-\overline{U'S'}) \right] \quad (\text{II-13})$$

Equation (II-13) may be interpreted as a convection-diffusion equation with a convective velocity  $U_f$  and a diffusive flux  $F$  defined as

$$F = -\overline{AE \frac{\partial S}{\partial x}} + \overline{A U' S'} \quad (\text{II-14})$$

In equation (II-14),  $\overline{A U' S'}$  represents the transport due to the oscillating tidal current. If it is assumed that a Fickian type diffusion process may approximate this type of mass transport, then

$$\overline{U' S'} = -E_{tA} \frac{\partial \overline{S}}{\partial x} \quad (\text{II-15})$$

where  $E_{tA}$  is the dispersion coefficient resulting from the tidal current.

The term  $-\overline{AE \frac{\partial S}{\partial x}}$  represents the time average of mass flux due to dispersion by the shear effect. To express this in terms of time averaged parameters, consider an estuarine river of width  $B$  and substitute equation (II-2) into the quantity  $-\overline{AE \frac{\partial S}{\partial x}}$ . This gives

$$-\overline{AE \frac{\partial S}{\partial x}} = -kBh^2 \overline{|U|} \frac{\partial \overline{S}}{\partial x} \quad (\text{II-16})$$

where  $B$  is assumed to be constant during a tidal cycle.

Meanwhile, the salt concentration  $S$ , the velocity  $U$ , and the depth  $h$  may be approximated by the following functions of space and time:

$$\begin{aligned}
S &= \bar{S}(x) + S_t(x) \sin \frac{2\pi}{T} t \\
U &= \bar{U}(x) + U_t(x) \sin \left( \frac{2\pi}{T} t + \theta \right) \\
h &= \bar{h}(x) + h_t(x) \sin \frac{2\pi}{T} t
\end{aligned} \tag{II-17}$$

where  $T$  is the tidal period. In the above approximations, the depth  $h$  is assumed to be in phase with the salt concentration but with a phase  $\theta$  relative to the depth-mean velocity. The following approximations were also employed to simplify the calculations:

$$\begin{aligned}
\frac{h_t}{\bar{h}} &\ll 1 \\
\frac{S_t}{\bar{S}} &\ll 1
\end{aligned} \tag{II-18}$$

$$\theta = \frac{\pi}{2} + \delta \text{ where } \delta \ll 1$$

Thus, to a first order approximation the term  $\overline{kh^2|U| \frac{\partial S}{\partial x}}$  reduces to

$$\begin{aligned}
\overline{kh^2|U| \frac{\partial S}{\partial x}} &= k \left\{ \left[ \frac{2}{\pi} \left( \sin^{-1} \frac{\bar{U}}{U_t} \right) \bar{h}^2 \bar{U} + \frac{2}{\pi} \sqrt{1 - \frac{\bar{U}^2}{U_t^2}} \bar{h}^2 U_t \right] \frac{\partial \bar{S}}{\partial x} \right\} \\
&= \bar{h} \bar{E} \frac{\partial \bar{S}}{\partial x}
\end{aligned} \tag{II-19}$$

where

$$\begin{aligned}
\bar{E} &= k \bar{h} \left[ \frac{2}{\pi} \left( \sin^{-1} \frac{\bar{U}}{U_t} \right) \bar{U} + \frac{2}{\pi} \sqrt{1 - \frac{\bar{U}^2}{U_t^2}} U_t \right] \\
&= k \bar{h} \overline{|U|}
\end{aligned} \tag{II-20}$$

The details of this analysis are presented in Appendix A1. Finally, equation (II-13) can be written as

$$\frac{\partial \bar{S}}{\partial t} + U_f \frac{\partial \bar{S}}{\partial x} = \frac{1}{A} \frac{\partial}{\partial x} \left[ \bar{A} E_A \frac{\partial \bar{S}}{\partial x} \right] \quad (\text{II-21})$$

where

$$E_A = \bar{E} + E_{tA}$$

and

$$E_{tA} = - \overline{U'S'} \left( \frac{\partial \bar{S}}{\partial x} \right)^{-1} \quad (\text{II-22})$$

In many practical applications of the salinity intrusion model, the significant information is the maximum or minimum salinity rather than the average value over a tidal cycle. Moreover, the field data of the average salinity distribution is tedious to collect. Most of the existing field data are the salinity distribution at high or low water slack. To formulate a model describing long-term variations of salinity at slack water, the instantaneous salt concentration can be represented in terms of the salt concentration at high water slack  $S_h$ , or low water slack  $S_L$ , by the following expressions:

$$S = S_h - S_h' \quad (\text{II-23})$$

$$S = S_L + S_L' \quad (\text{II-24})$$

where  $S_h' > 0$ ,  $S_L' > 0$ ,  $\overline{S_h'} \neq 0$ , and  $\overline{S_L'} \neq 0$ .

An analysis, similar to that employed for the tidal averaged model where  $S = \bar{S} + S'$ , is performed for the slack tide approximations to determine the form of the dispersion coefficients  $E_h$  and  $E_L$ . The details of



this analysis are presented in Appendix A2. In the calculations, Reynolds axioms were assumed valid,  $S_h$  and  $S_L$  were considered constant during a tidal cycle, and finally  $S_h' \ll S_h$  and  $S_L' \ll S_L$ .\*

The dispersion coefficients for high water slack  $E_h$  and low water slack  $E_L$  were determined as follows:

$$E_h = \bar{E} + E_{th} \quad (\text{II-25})$$

$$E_L = \bar{E} + E_{tL} \quad (\text{II-26})$$

where  $\bar{E} = k \bar{h} \overline{|U|}$

$$E_{th} = \overline{U'S_h'} \left( \frac{\partial S_h}{\partial x} \right)^{-1} \quad (\text{II-27})$$

$$E_{tL} = - \overline{U'S_L'} \left( \frac{\partial S_L}{\partial x} \right)^{-1} \quad (\text{II-28})$$

---

\*

Since  $S_L < S_h$ , the assumption  $S_L' \ll S_L$  is not as strong as the assumption  $S_h' \ll S_h$ , and both of these assumptions are not valid for upstream sections of the estuarine river where the salinity values are low.

### III

#### CASE STUDY OF THE RAPPAHANNOCK RIVER, VIRGINIA

#### Theoretical and Experimental Determination of the Dispersion Coefficient

A tidal estuary is a challenging physical system in which the tidal motion produces an oscillating flow. As had been mentioned in the Introduction, the dispersion resulting from shear effect in oscillating flow depends on the dimensionless parameter

$$T' = \frac{T}{T_c} \quad (\text{III-1})$$

where  $T$  is the period of oscillation and  $T_c$  is the time scale for cross-sectional mixing.  $T_c$  can be represented by the following expression:

$$T_c = \frac{L^2}{e} \quad (\text{III-1a})$$

where  $L$  is the length parameter of the cross-section and  $e$  is the turbulent diffusivity.

The shear effect may be produced by velocity gradients in the vertical and transverse directions. Holley et. al. (1970) defined  $T'$  for the transverse and vertical directions as follows:

$$T_t' = T \left[ \frac{\left(\frac{B}{2}\right)^2}{e_z} \right]^{-1} \quad (\text{III-2})$$

$$T_v' = T \left[ \frac{h^2}{e_y} \right]^{-1} \quad (\text{III-3})$$

where

$T$  = tidal period

$B$  = width

$h$  = depth

$e_z$  = vertical turbulent diffusivity

$e_y$  = transverse turbulent diffusivity

They approximated  $e_z$  and  $e_y$  by the following expressions:

$$e_y = 0.067Ru_* \quad (\text{III-4})$$

$$e_z = 0.23Ru_* \quad (\text{III-5})$$

where

$R$  = hydraulic radius

$u_*$  = shear velocity corresponding to the tidal velocity averaged over one-half period

The expression for  $e_y$  was obtained assuming a logarithmic velocity profile and a von Karman constant of 0.4, while the expression for  $e_z$  is empirical for a straight uniform channel.

In this case study,  $T_t'$  and  $T_v'$  were calculated for various transects along the Rappahannock River, Virginia, using equations (III-2) and (III-3). The depth  $h$  and the hydraulic radius  $R$  were assumed equal and the shear velocity

$u_*$  equaled  $3.9 \left[ \frac{n}{R^{1/6}} \right] \left[ \frac{2U_t}{\pi} \right]$  where  $n$  is the Manning friction coefficient,  $U_t$  is the amplitude of the tidal velocity in the units of feet per second and  $R$  is the hydraulic radius in the units of feet. The values for  $T_t'$ ,  $T_v'$  and the hydraulic parameters are presented in Table 1. Since  $T_v' \gg 1$  and  $T_t' \ll 1$ , the vertical shear is the primary dispersive mechanism in this estuary and the dispersion coefficient can be approximated by Elder's (1959) formulation

$$E = E_v = 6hu_* \quad (\text{III-6})$$

The modified Taylor equation

$$E = 77nUR^{5/6} \quad (\text{in ft-sec units}) \quad (\text{III-7})$$

is more frequently used to estimate the dispersion coefficients for unidirectional flows in straight open channels where  $U$  is the cross-sectional mean velocity. For oscillating flows in estuaries, Harleman (1971) suggested that the modified Taylor equation (III-7) be increased by a factor of two to account for the bends and irregularities of the channel. He proposed that

$$\bar{E} = 100nU_tR^{5/6} \quad (\text{in ft-sec units}) \quad (\text{III-8})$$

where  $U_t$  is the tidal amplitude. Since the longitudinal dispersion coefficient is usually expressed in the units of square miles per day, equation (III-8) can be transformed to the expression

$$\bar{E} = 0.3nU_tR^{5/6} \left[ \begin{array}{l} (E \text{ in sq.mi/day}) \\ (U_t \text{ and } R \text{ in ft-sec}) \end{array} \right] \quad (\text{III-9})$$

Table 1. Hydraulic Parameters and Dispersion Coefficients along sections of the Rappahannock River, Virginia

Distance from River Mouth (miles)	Hydraulic Radius R (ft)	$U_t$ (ft/sec)	Width B (ft)	$T_v^*$	$T_t^*$ ( $10^{-2}$ )	$\bar{E}_v = 6hu^*$ ( $10^{-2}$ ) (sq.mi/day)	$\bar{E}^+ = 0.3nU_tR^{5/6}$ ( $10^{-2}$ ) (sq.mi/day)
105.9	9.2	0.23	390	3.59	2.73	0.19	1.23
89.6	8.3	0.72	785	12.63	1.93	0.55	3.53
77.9	6.5	1.25	2300	29.25	0.32	0.77	4.99
68.8	30.0	1.48	600	5.67	19.9	3.28	21.2
57.5	12.2	1.28	3200	14.37	0.29	1.34	8.64
49.3	14.6	1.15	3350	10.43	0.27	1.40	9.03

\*  $T = 12.4$  hours and  $n = 0.028$

$^+ \text{Mean } \bar{E} = 8.103 \times 10^{-2}$  sq.mi/day, the average dispersion coefficient due to shear effect

Finally, equations (III-6) and (III-9) are used to calculate the dispersion coefficients resulting from the shear effect for transects of the Rappahannock River and these results are tabulated in Table 1.

The Physical Oceanography and Hydraulics Department of the Virginia Institute of Marine Science conducted dye dispersion tests in the Rappahannock River, Virginia during July and August 1970. Rhodamine dye was discharged into the river at the Rappahannock River Bridge near Fredericksburg, 106.3 miles from the river mouth, on 8 July 1970. Dye concentration observations were made at high or low water slack from July 11 through August 18, for a 90 mile region of the river with an upper reach, 125 miles from the river mouth.

If the time history of the dye distribution at low water slack may be accurately described by the slack tide approximation model, the convection-diffusion equation (A2-22) suggests that the dye distribution will tend to become normally distributed with the dispersion coefficient

$$E_L = \frac{1}{2} \frac{d}{dt} \sigma^2 \quad (\text{III-10})$$

where  $\sigma^2(t)$  is the variance of the distribution. This expression (III-10) is used to calculate the dispersion coefficient for the Rappahannock River field data. The procedure is described in the following paragraphs.

First, the dye concentration for each date was plotted as a function of distance from the river mouth.

Figure 1 shows this graph for 4 August 1970. Near the tail of the graph, the curve was extrapolated to zero. The accumulative area under the curve, as a function of distance from the river mouth, was calculated numerically using Simpson's Rule, contained in an IBM subroutine package.

To test the normality of the dye distribution, the normalized accumulative areas were plotted against distance from the river mouth on normal probability paper. Figure 2 and Figure 3 show two samples of these plots. It is observed that after an initial period from the time the dye was released, the dye distribution approached a normal distribution.

In the normal paper plots, the best straight lines were drawn through the points and the difference between the distance at the 16th and 84th percentile equaled twice the standard deviation  $\sigma$ . To obtain the quantity  $\frac{d}{dt} \sigma^2$ , the variance  $\sigma^2$  was plotted against time and is shown in Figure 4. The best straight line was drawn through the points and the slope  $\frac{d}{dt} \sigma^2$  was calculated to be 3.9 sq.mi/day. The dispersion coefficient was approximated by equation (III-10) and equals 1.95 sq.mi/day.

This observed dispersion coefficient consists of dispersion due to the shear effect and the oscillating tidal currents. This experimental value of the dispersion coefficient is greater than those listed in Table 1 by two

orders of magnitude. The difference between these two coefficients is an estimate of the magnitude of  $E_{tL}$  .

Thus

$$E_{tL} \approx 1.95 - 8.1 \times 10^{-2} \gg \bar{E}$$

where  $\bar{E} = 8.1 \times 10^{-2}$  .



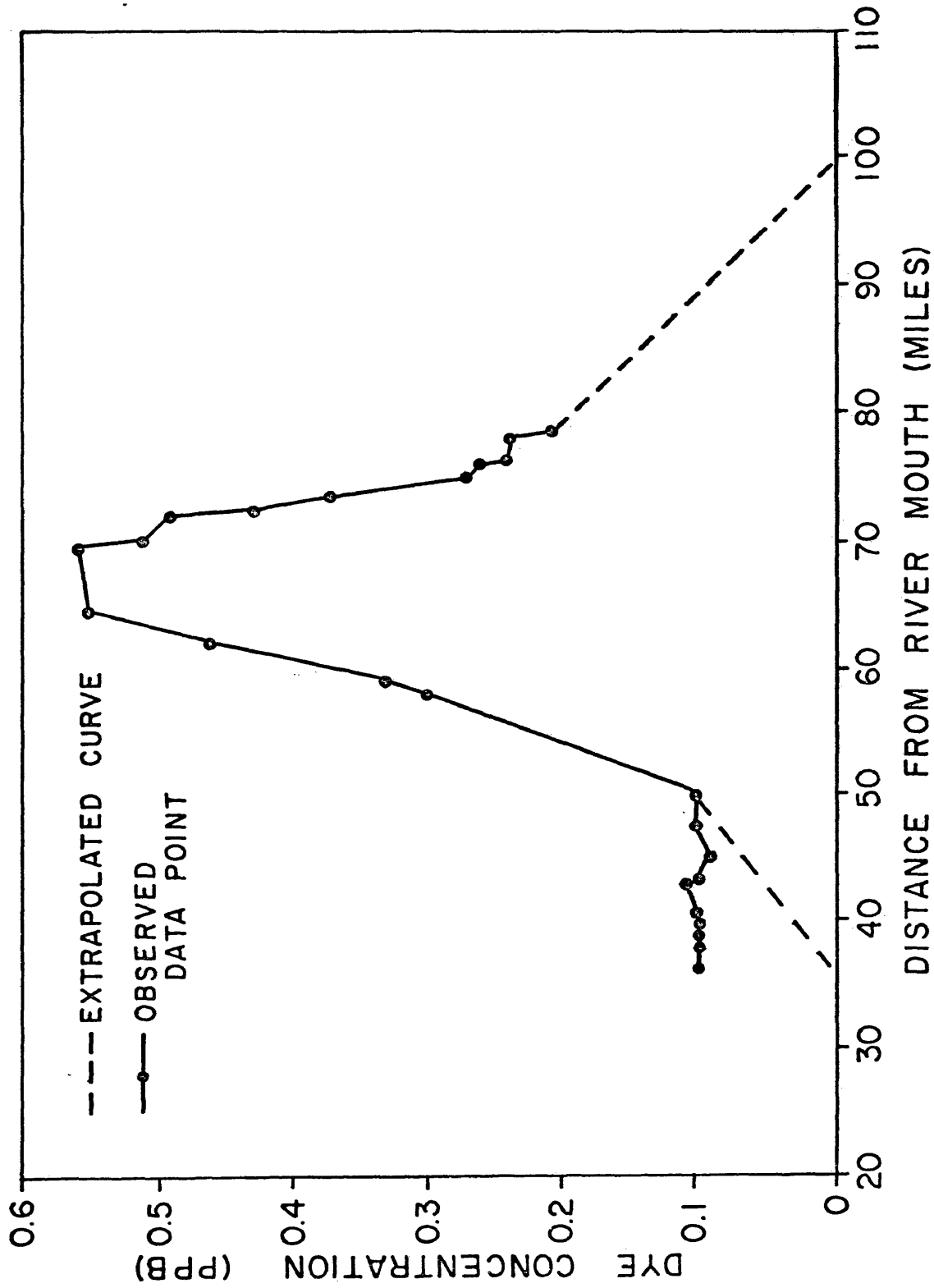


Figure 1. Dye concentration at low water slack as a function of distance from the river mouth along the Rappahannock River, Virginia; 4 August 1970.

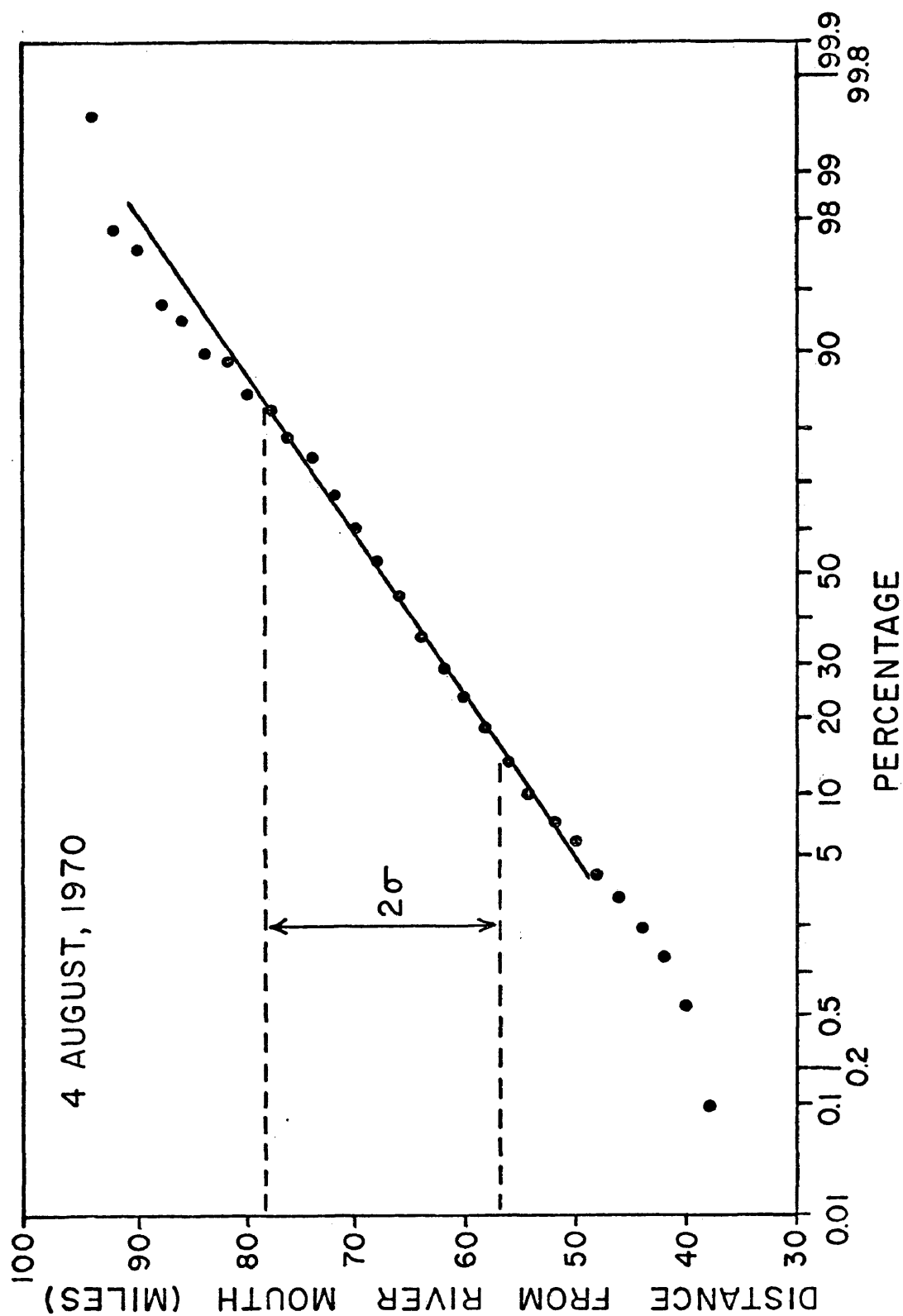


Figure 2. Normalized accumulative areas of the dye distribution at low water slack as a function of distance from the river mouth along the Rappahannock River, Virginia; 4 August 1970.

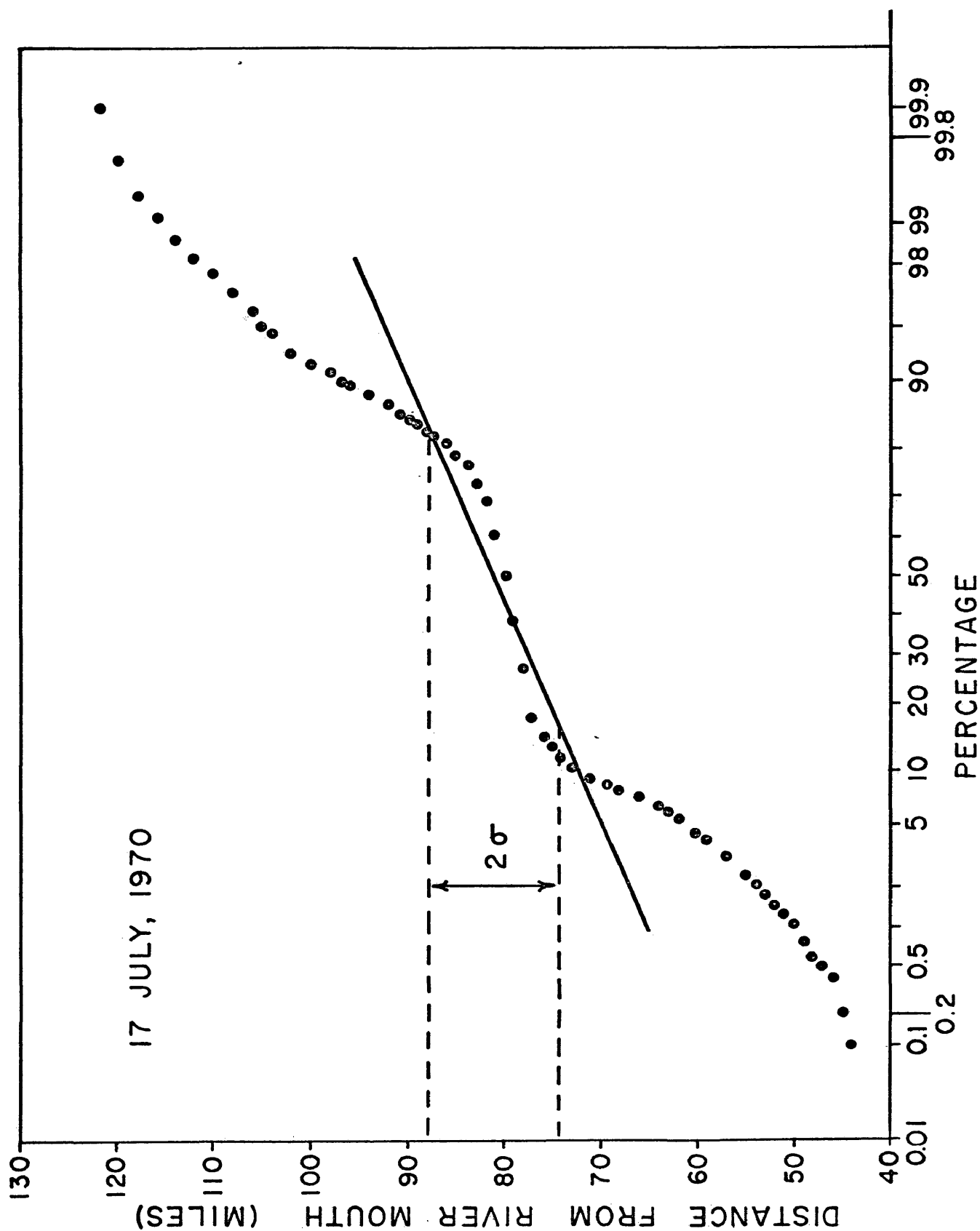


Figure 3. Normalized accumulative areas of the dye distribution at low water slack as a function of distance from the river mouth along the Rappahannock River, Virginia; 17 July, 1970.

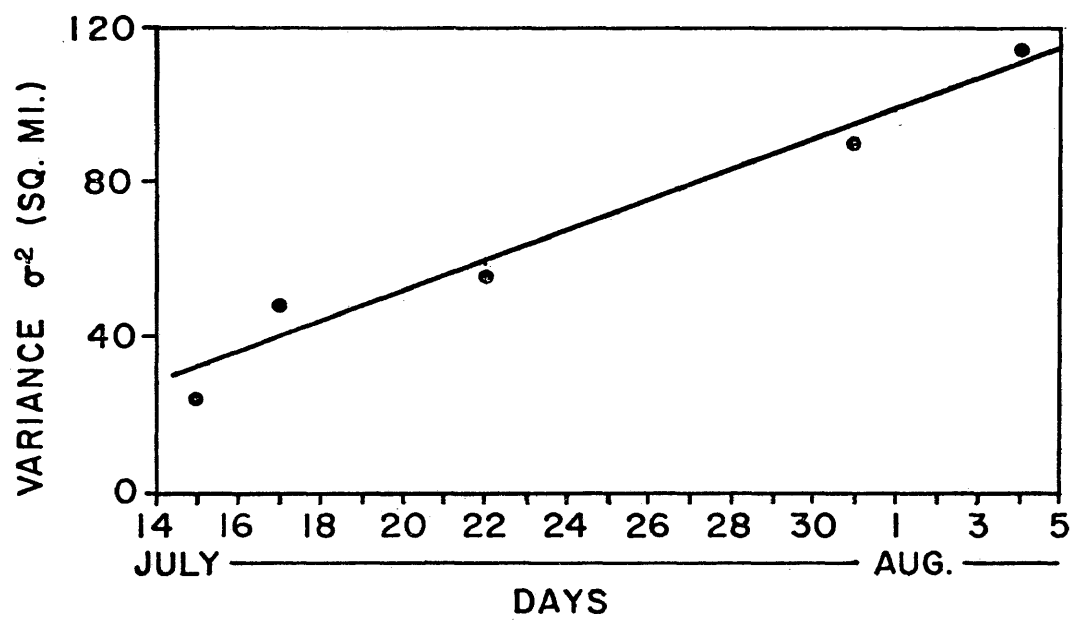


Figure 4. Variance of the dye distribution at low water slack as a function of time.

#### IV

##### DISCUSSION AND CONCLUSIONS

The salinity distribution along an estuarine river can be described approximately by the one-dimensional equation

$$\frac{\partial S^*}{\partial t} + U_f \frac{\partial S^*}{\partial x} = \frac{1}{A} \frac{\partial}{\partial x} \left[ E^{*A} \frac{\partial S^*}{\partial x} \right] \quad (\text{IV-1})$$

where (1)  $E^* = \bar{E} + E_{tA}$ , if  $S^* = \bar{S}$ , the average salinity over a tidal cycle;

(2)  $E^* = \bar{E} + E_{th}$ , if  $S^* = S_h$ , the salinity at high water slack;

(3)  $E^* = \bar{E} + E_{tL}$ , if  $S^* = S_L$ , the salinity at low water slack.

The component  $\bar{E}$  is derived from the contribution of velocity shear to the dispersion mechanism and is identical for all three models with  $\bar{E} = kh \overline{|U|}$ . The component  $E_t^*$  models the mass transport by the oscillating tidal currents and is defined as follows:

$$E_t^* = E_{tA} = - \overline{U'S'} \left[ \frac{\partial \bar{S}}{\partial x} \right]^{-1} \quad (\text{IV-2})$$

or

$$E_t^* = E_{th} = \overline{U'S_h'} \left[ \frac{\partial S_h}{\partial x} \right]^{-1} \quad (\text{IV-3})$$

or

$$E_t^* = E_{tL} = - \overline{U'S_L'} \left[ \frac{\partial S_L}{\partial x} \right]^{-1} \quad (\text{IV-4})$$

The analysis of dye dispersion studies in the Rappahannock River, Virginia revealed that  $E_t^* \gg \bar{E}$  where  $\bar{E}$  was calculated using the modified-Taylor equation. The observed value of  $E^*$  was calculated to be 1.95 sq.mi/day compared to  $\bar{E} \approx 10^{-2}$  sq.mi/day; for a long-term model, therefore, the dispersion due to the oscillating tidal currents is more important than the dispersion due to the shear effect.

If the temporal variations of the velocity and salt concentration,  $U'$  and  $S_*'$  ( $S_*' = S'$ , or  $S_h'$  or  $S_L'$ ) are approximated by a sinusoidal function of time, the quantity  $\overline{U'S_*'} = \pm \frac{U_t S_t}{2} \cos\theta$ , for all three models. The calculations are presented in Appendix A3. The difference between the dispersion coefficients of the three long-term models, therefore, is dependent upon the salinity gradients  $\frac{\partial \bar{S}}{\partial x}$ ,  $\frac{\partial S_h}{\partial x}$ , and  $\frac{\partial S_L}{\partial x}$ .

Harleman (1971) calculated the dispersion coefficients from the field and laboratory data of salinity distributions and indicated that  $E_{th} > E_{tL}$  in the lowest one-fifth portion of estuarine rivers. If equations (IV-3) and (IV-4) are valid, then  $\frac{\partial S_h}{\partial x} < \frac{\partial S_L}{\partial x}$ . To check the validity of this conclusion, slack water run data from the York River, Virginia was examined. Observations during the latter part of May through September 1971 were chosen because of less stratification of the water column during these months.

The data is presented in Figures 5 and 6. These graphs show that for salinity observations at distances greater than five miles from the river mouth, the slope of the high water slack salinity data is less than that for low water slack. This data agrees with the statement

$$\frac{\partial S_L}{\partial x} > \frac{\partial S_h}{\partial x}, \text{ and thus } E_{th} > E_{tL}.$$

Before equation (IV-1) may be introduced into practical application to solve for the salinity distribution, the dispersion coefficient  $E^*$  must be estimated from bulk hydraulic data. At present, no rigorous theory about the formulation of  $E_t^*$  has yet been developed and further research should be done in this area.

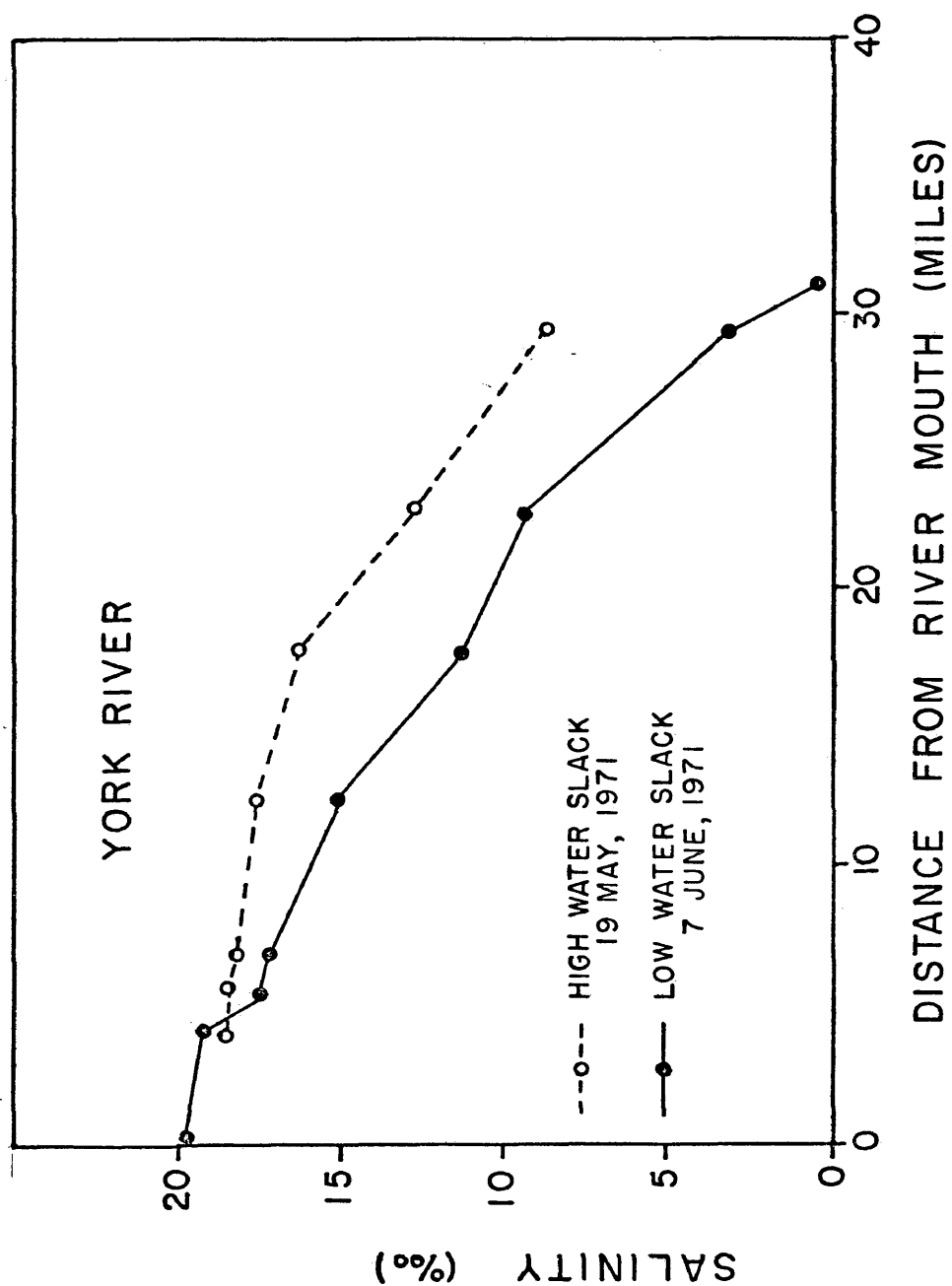


Figure 5. Salinity distribution at high and low water slack along the York River, Virginia; 19 May 1971, 7 June 1971.



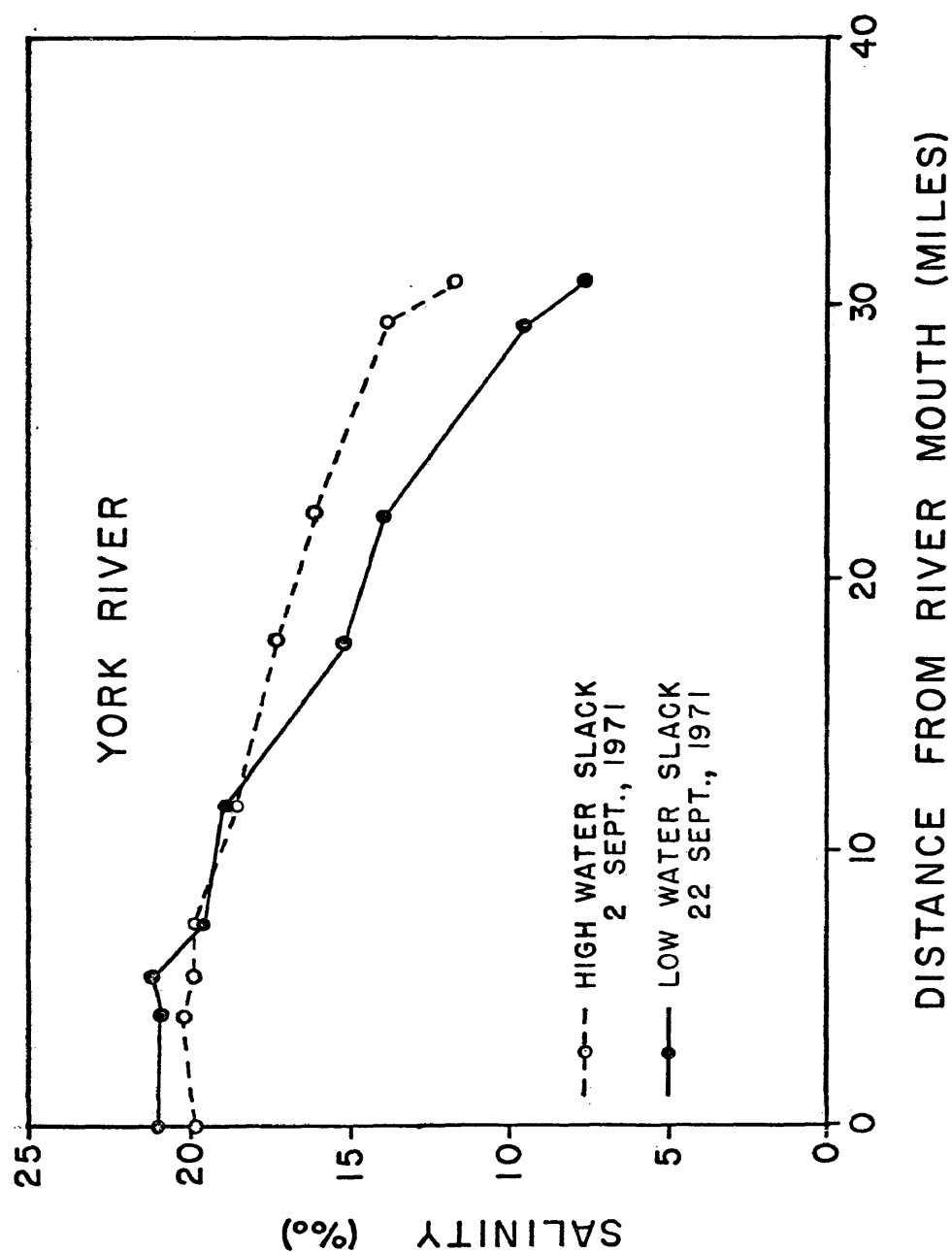


Figure 6. Salinity distribution at high and low water slack along the York River, Virginia; 2 September 1971, 22 September 1971.

APPENDIX A1  
AVERAGE MASS TRANSPORT BY DISPERSION FROM  
THE SHEAR EFFECT

The average dispersive mass transport across a transect of estuary is given by equation (II-16) as

$$\overline{-AE \frac{\partial S}{\partial x}} = -k \overline{Bh^2 |U| \frac{\partial S}{\partial x}} \quad (A1-1)$$

where the average operation over a tidal cycle is defined as

$$\overline{h^2 |U| \frac{\partial S}{\partial x}} = \frac{1}{T} \int_{-T/2}^{T/2} h^2 |U| \frac{\partial S}{\partial x} dt \quad (A1-2)$$

Since the integrand consists of a factor  $|U|$ , it is necessary to consider two separate cases: one in which  $\bar{U} \geq U_t$ , the other in which  $\bar{U} < U_t$ .

First, let us consider the case where  $\bar{U} \geq U_t$  and  $U$  is defined by equation (II-17), then

$$|U| = U \quad (A1-3)$$

Next substitute expression (A1-3) into equation (A1-2).

This gives

$$\overline{h^2 |U| \frac{\partial S}{\partial x}} = \frac{1}{T} \int_{-T/2}^{T/2} \left[ h^2 U \frac{\partial S}{\partial x} \right] dt \quad (A1-4)$$

To evaluate equation (A1-4), substitute equation (II-17)

into it and we have

$$\overline{h^2 |U| \frac{\partial S}{\partial x}} = \frac{1}{T} \int_{-T/2}^{T/2} (\bar{h} + h_t \sin \frac{2\pi}{T} t)^2 dt \quad (A1-5)$$

$$(\bar{U} + U_t \sin (\frac{2\pi}{T} t + \theta)) \frac{\partial}{\partial x} (\bar{S} + S_t \sin \frac{2\pi}{T} t) dt$$

Now let  $m = -\frac{T}{2}$ ,  $n = \frac{T}{2}$ , and expand the integrand of equation (A1-5). Thus, we obtain

$$\begin{aligned} \overline{h^2 U \frac{\partial S}{\partial x}} &= \frac{1}{T} \int_m^n \bar{h}^2 \bar{U} \frac{\partial \bar{S}}{\partial x} dt \\ &+ \frac{1}{T} \int_m^n \bar{h}^2 \bar{U} \frac{\partial S_t}{\partial x} \sin \frac{2\pi}{T} t dt \\ &+ \frac{1}{T} \int_m^n \bar{h}^2 U_t \frac{\partial \bar{S}}{\partial x} \sin (\frac{2\pi}{T} t + \theta) dt \\ &+ \frac{1}{T} \int_m^n \bar{h}^2 U_t \frac{\partial S_t}{\partial x} \sin \frac{2\pi}{T} t \sin (\frac{2\pi}{T} t + \theta) dt \\ &+ \frac{1}{T} \int_m^n 2h_t \bar{h} \bar{U} \frac{\partial \bar{S}}{\partial x} \sin \frac{2\pi}{T} t dt \\ &+ \frac{1}{T} \int_m^n 2h_t \bar{h} \bar{U} \frac{\partial S_t}{\partial x} \sin^2 \frac{2\pi}{T} t dt \\ &+ \frac{1}{T} \int_m^n 2h_t \bar{h} U_t \frac{\partial \bar{S}}{\partial x} \sin \frac{2\pi}{T} t \sin(\frac{2\pi}{T} t + \theta) dt \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{T} \int_m^n 2h_t \bar{h} U_t \frac{\partial S_t}{\partial x} \sin^2 \frac{2\pi}{T} t \sin \left( \frac{2\pi}{T} t + \theta \right) dt \\
& + \frac{1}{T} \int_m^n h_t^2 \bar{U} \frac{\partial \bar{S}}{\partial x} \sin^2 \frac{2\pi}{T} t dt \\
& + \frac{1}{T} \int_m^n h_t^2 \bar{U} \frac{\partial S_t}{\partial x} \sin^3 \frac{2\pi}{T} t dt \\
& + \frac{1}{T} \int_m^n h_t^2 U_t \frac{\partial \bar{S}}{\partial x} \sin^2 \frac{2\pi}{T} t \sin \left( \frac{2\pi}{T} t + \theta \right) dt \\
& + \frac{1}{T} \int_m^n h_t^2 U_t \frac{\partial S_t}{\partial x} \sin^3 \frac{2\pi}{T} t \sin \left( \frac{2\pi}{T} t + \theta \right) dt
\end{aligned} \tag{A1-6}$$

Next, let  $\phi = \frac{2\pi}{T} t$ , then

$$dt = \frac{T}{2\pi} d\phi$$

and  $\sin(\phi + \theta) = \sin\phi \cos\theta + \cos\phi \sin\theta$

The above relationships are used to evaluate the integrals and simplify equation (A1-6). We then find that

$$\begin{aligned}
& \overline{h^2 U \frac{\partial S}{\partial x}} \\
& = \bar{h}^2 \bar{U} \frac{\partial \bar{S}}{\partial x} \left[ \text{Integral 1} \right] \\
& + \bar{h}^2 \bar{U} \frac{\partial S_t}{\partial x} \left[ \text{Integral 2} \right] \\
& + \bar{h}^2 U_t \frac{\partial \bar{S}}{\partial x} \cos\theta \left[ \text{Integral 2} \right]
\end{aligned}$$

$$\begin{aligned}
& + \bar{h}^2 U_t \frac{\partial \bar{S}}{\partial x} \sin \theta \left[ \text{Integral } 8 \right] \\
& + \bar{h}^2 U_t \frac{\partial S_t}{\partial x} \cos \theta \left[ \text{Integral } 4 \right] \\
& + \bar{h}^2 U_t \frac{\partial S_t}{\partial x} \sin \theta \left[ \text{Integral } 3 \right] \\
& + 2h_t \bar{h} \bar{U} \frac{\partial \bar{S}}{\partial x} \left[ \text{Integral } 2 \right] \\
& + 2h_t \bar{h} \bar{U} \frac{\partial S_t}{\partial x} \left[ \text{Integral } 4 \right] \\
& + 2h_t \bar{h} U_t \frac{\partial \bar{S}}{\partial x} \cos \theta \left[ \text{Integral } 4 \right] \\
& + 2h_t \bar{h} U_t \frac{\partial \bar{S}}{\partial x} \sin \theta \left[ \text{Integral } 3 \right] \\
& + 2h_t \bar{h} U_t \frac{\partial S_t}{\partial x} \cos \theta \left[ \text{Integral } 5 \right] \\
& + 2h_t \bar{h} U_t \frac{\partial S_t}{\partial x} \sin \theta \left[ \text{Integral } 6 \right] \\
& + h_t^2 \bar{U} \frac{\partial \bar{S}}{\partial x} \left[ \text{Integral } 4 \right] \\
& + h_t^2 \bar{U} \frac{\partial S_t}{\partial x} \left[ \text{Integral } 5 \right] \\
& + h_t^2 U_t \frac{\partial \bar{S}}{\partial x} \cos \theta \left[ \text{Integral } 5 \right] \\
& + h_t^2 U_t \frac{\partial \bar{S}}{\partial x} \sin \theta \left[ \text{Integral } 6 \right] \\
& + h_t^2 U_t \frac{\partial S_t}{\partial x} \cos \theta \left[ \text{Integral } 7 \right] \\
& + h_t^2 U_t \frac{\partial S_t}{\partial x} \sin \theta \left[ \text{Integral } 9 \right]
\end{aligned}
\tag{A1-7}$$

where

$$\text{Integral 1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi = 1$$

$$\text{Integral 2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin\phi \, d\phi = 0$$

$$\text{Integral 3} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin\phi \cos\phi \, d\phi = 0$$

$$\text{Integral 4} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^2\phi \, d\phi = \frac{1}{2}$$

$$\text{Integral 5} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^3\phi \, d\phi = 0$$

$$\text{Integral 6} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^2\phi \cos\phi \, d\phi = 0$$

$$\text{Integral 7} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^4\phi \, d\phi = \frac{3}{8}$$

$$\text{Integral 8} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos\phi \, d\phi = 0$$

$$\text{Integral 9} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^3\phi \cos\phi \, d\phi = 0$$

So, equation (A1-7) simplifies to

$$\begin{aligned} \overline{h^2 U \frac{\partial \bar{S}}{\partial x}} &= \bar{h}^2 \bar{U} \frac{\partial \bar{S}}{\partial x} + \bar{h}^2 U_t \frac{\partial S_t}{\partial x} \frac{\cos\theta}{2} \\ &+ h_t \bar{h} \bar{U} \frac{\partial S_t}{\partial x} + h_t \bar{h} U_t \frac{\partial \bar{S}}{\partial x} \cos\theta \\ &+ \frac{1}{2} h_t^2 \bar{U} \frac{\partial \bar{S}}{\partial x} + \frac{3}{8} h_t^2 U_t \frac{\partial S_t}{\partial x} \cos\theta \end{aligned} \quad (\text{A1-8})$$

Equation (A1-8) can be further simplified by using the following approximations:

$$\begin{aligned}\frac{h_t}{\bar{h}} &<< 1 \\ \frac{s_t}{\bar{s}} &<< 1\end{aligned}\tag{A1-8a}$$

$$\theta = \frac{\pi}{2} + \delta \text{ where } \delta << 1$$

Thus, to a first order approximation equation (A1-8) reduces to

$$\overline{h^2 U \frac{\partial \bar{S}}{\partial x}} \approx \bar{h}^2 \bar{U} \frac{\partial \bar{S}}{\partial x}\tag{A1-9}$$

Therefore, the dispersion coefficient  $\bar{E}$  is determined from equations (A1-1) and (A1-9), such that

$$\begin{aligned}-k B \overline{h^2 |U| \frac{\partial \bar{S}}{\partial x}} &= -k B \bar{h}^2 \bar{U} \frac{\partial \bar{S}}{\partial x} \\ &= -k \bar{A} \bar{h} \bar{U} \frac{\partial \bar{S}}{\partial x} \\ &= -\bar{A} \bar{E} \frac{\partial \bar{S}}{\partial x}\end{aligned}\tag{A1-10}$$

and  $\bar{E} = k \bar{h} \bar{U}$ .

The second case which deserves consideration is when  $\bar{U} < U_t$ , then

$$|U| = U = \bar{U} + U_t \sin \left( \frac{2\pi}{T} t + \theta \right)\tag{A1-11}$$

if

$$-\sin^{-1} \frac{\bar{U}}{U_t} - \theta \leq \frac{2\pi}{T} t \leq \pi + \sin^{-1} \frac{\bar{U}}{U_t} - \theta$$

$$\text{and } |U| = -U = -\bar{U} - U_t \sin \left( \frac{2\pi}{T} t + \theta \right) \quad (\text{A1-12})$$

if

$$\pi + \sin^{-1} \frac{\bar{U}}{U_t} - \theta \leq \frac{2\pi}{T} t \leq 2\pi - \sin^{-1} \frac{\bar{U}}{U_t} - \theta$$

Thus, the average operation is divided into two integrals where

$$\begin{aligned} \overline{h^2 |U| \frac{\partial S}{\partial x}} &= \frac{1}{T} \int_{f(T/2\pi)}^{g(T/2\pi)} h^2 U \frac{\partial S}{\partial x} dt \\ &- \frac{1}{T} \int_{g(T/2\pi)}^{j(T/2\pi)} h^2 U \frac{\partial S}{\partial x} dt \end{aligned} \quad (\text{A1-13})$$

$$\text{and } f = -\sin^{-1} \frac{\bar{U}}{U_t} - \theta$$

$$g = \pi + \sin^{-1} \frac{\bar{U}}{U_t} - \theta$$

$$j = 2\pi - \sin^{-1} \frac{\bar{U}}{U_t} - \theta$$

$T$  = tidal period

Before the integrals can be evaluated, equations (II-17) are substituted into equation (A1-13). This gives

$$\begin{aligned} \overline{h^2 |U| \frac{\partial S}{\partial x}} &= \frac{1}{T} \int_{f(T/2\pi)}^{g(T/2\pi)} h^2 \left[ \bar{U} + U_t \sin \left( \frac{2\pi}{T} t + \theta \right) \right] \frac{\partial \bar{S}}{\partial x} dt \end{aligned}$$



$$\begin{aligned}
& - \frac{1}{T} \int_{g(T/2\pi)}^{j(T/2\pi)} \bar{h}^2 \left[ \bar{U} + U_t \sin \left( \frac{2\pi}{T} t + \theta \right) \right] \frac{\partial \bar{S}}{\partial x} dt \\
& + \frac{1}{T} \int_{f(T/2\pi)}^{g(T/2\pi)} \bar{h}^2 \left[ \bar{U} + U_t \sin \left( \frac{2\pi}{T} t + \theta \right) \right] \sin \frac{2\pi}{T} t \frac{\partial S_t}{\partial x} dt \\
& - \frac{1}{T} \int_{g(T/2\pi)}^{j(T/2\pi)} \bar{h}^2 \left[ \bar{U} + U_t \sin \left( \frac{2\pi}{T} t + \theta \right) \right] \sin \frac{2\pi}{T} t \frac{\partial S_t}{\partial x} dt \\
& + \frac{1}{T} \int_{f(T/2\pi)}^{g(T/2\pi)} 2\bar{h} h_t \sin \frac{2\pi}{T} t \left[ \bar{U} + U_t \sin \left( \frac{2\pi}{T} t + \theta \right) \right] \frac{\partial \bar{S}}{\partial x} dt \\
& - \frac{1}{T} \int_{g(T/2\pi)}^{j(T/2\pi)} 2\bar{h} h_t \sin \frac{2\pi}{T} t \left[ \bar{U} + U_t \sin \left( \frac{2\pi}{T} t + \theta \right) \right] \frac{\partial \bar{S}}{\partial x} dt \\
& + \frac{1}{T} \int_{f(T/2\pi)}^{g(T/2\pi)} 2\bar{h} h_t \sin \frac{2\pi}{T} t \left[ \bar{U} + U_t \sin \left( \frac{2\pi}{T} t + \theta \right) \right] \sin \frac{2\pi}{T} t \frac{\partial S_t}{\partial x} dt \\
& - \frac{1}{T} \int_{g(T/2\pi)}^{j(T/2\pi)} 2\bar{h} h_t \sin \frac{2\pi}{T} t \left[ \bar{U} + U_t \sin \left( \frac{2\pi}{T} t + \theta \right) \right] \sin \frac{2\pi}{T} t \frac{\partial S_t}{\partial x} dt \\
& + \frac{1}{T} \int_{f(T/2\pi)}^{g(T/2\pi)} h_t^2 \sin^2 \frac{2\pi}{T} t \left[ \bar{U} + U_t \sin \left( \frac{2\pi}{T} t + \theta \right) \right] \frac{\partial \bar{S}}{\partial x} dt \\
& - \frac{1}{T} \int_{g(T/2\pi)}^{j(T/2\pi)} h_t^2 \sin^2 \frac{2\pi}{T} t \left[ \bar{U} + U_t \sin \left( \frac{2\pi}{T} t + \theta \right) \right] \frac{\partial \bar{S}}{\partial x} dt \\
& + \frac{1}{T} \int_{f(T/2\pi)}^{g(T/2\pi)} h_t^2 \sin^2 \frac{2\pi}{T} t \left[ \bar{U} + U_t \sin \left( \frac{2\pi}{T} t + \theta \right) \right] \sin \frac{2\pi}{T} t \frac{\partial S_t}{\partial x} dt
\end{aligned}$$

$$- \frac{1}{T} \int_{g(T/2\pi)}^{j(T/2\pi)} h_t^2 \sin^2 \frac{2\pi}{T} t \left[ \bar{U} + U_t \sin \left( \frac{2\pi}{T} t + \theta \right) \right] \sin \frac{2\pi}{T} t \frac{\partial S_t}{\partial x} dt$$

(A1-14)

Now let  $\phi = \frac{2\pi}{T} t$  in equation (A1-14), then

$$dt = \frac{T}{2\pi} d\phi$$

and equation (A1-14) can be rewritten as

$$\begin{aligned} h^2 |U| \frac{\partial \bar{S}}{\partial x} = & \left[ \bar{h}^2 \bar{U} \frac{\partial \bar{S}}{\partial x} \right] \frac{1}{2\pi} \left[ \int_f^g d\phi - \int_g^j d\phi \right] \\ & + \left[ \bar{h}^2 U_t \frac{\partial \bar{S}}{\partial x} \right] \frac{1}{2\pi} \left[ \int_f^g \sin(\phi + \theta) d\phi - \int_g^j \sin(\phi + \theta) d\phi \right] \\ & + \left[ \bar{h}^2 \bar{U} \frac{\partial S_t}{\partial x} \right] \frac{1}{2\pi} \left[ \int_f^g \sin \phi d\phi - \int_g^j \sin \phi d\phi \right] \\ & + \left[ \bar{h}^2 U_t \frac{\partial S_t}{\partial x} \right] \frac{1}{2\pi} \left[ \int_f^g \sin(\phi + \theta) \sin \phi d\phi - \int_g^j \sin(\phi + \theta) \sin \phi d\phi \right] \\ & + \left[ 2\bar{h} h_t \bar{U} \frac{\partial \bar{S}}{\partial x} \right] \frac{1}{2\pi} \left[ \int_f^g \sin \phi d\phi - \int_g^j \sin \phi d\phi \right] \\ & + \left[ 2\bar{h} h_t U_t \frac{\partial \bar{S}}{\partial x} \right] \frac{1}{2\pi} \left[ \int_f^g \sin(\phi + \theta) \sin \phi d\phi - \int_g^j \sin(\phi + \theta) \sin \phi d\phi \right] \\ & + \left[ 2\bar{h} h_t \bar{U} \frac{\partial S_t}{\partial x} \right] \frac{1}{2\pi} \left[ \int_f^g \sin^2 \phi d\phi - \int_g^j \sin^2 \phi d\phi \right] \\ & + \left[ 2\bar{h} h_t U_t \frac{\partial S_t}{\partial x} \right] \frac{1}{2\pi} \left[ \int_f^g \sin^2 \phi \sin(\phi + \theta) d\phi - \int_g^j \sin^2 \phi \sin(\phi + \theta) d\phi \right] \end{aligned}$$

$$\begin{aligned}
& + \left[ h_t^2 \bar{U} \frac{\partial \bar{S}}{\partial x} \right] \frac{1}{2\pi} \left[ \int_f^g \sin^2 \phi d\phi - \int_g^j \sin^2 \phi d\phi \right] \\
& + \left[ h_t^2 U_t \frac{\partial \bar{S}}{\partial x} \right] \frac{1}{2\pi} \left[ \int_f^g \sin^2 \phi \sin(\phi+\theta) d\phi - \int_g^j \sin^2 \phi \sin(\phi+\theta) d\phi \right] \\
& + \left[ h_t^2 \bar{U} \frac{\partial S_t}{\partial x} \right] \frac{1}{2\pi} \left[ \int_f^g \sin^3 \phi d\phi - \int_g^j \sin^3 \phi d\phi \right] \\
& + \left[ h_t^2 U_t \frac{\partial S_t}{\partial x} \right] \frac{1}{2\pi} \left[ \int_f^g \sin^3 \phi \sin(\phi+\theta) d\phi - \int_g^j \sin^3 \phi \sin(\phi+\theta) d\phi \right]
\end{aligned}$$

(A1-15)

But  $\sin(\phi+\theta) = \sin\phi \cos\theta + \cos\phi \sin\theta$ , so equation

(A1-15) becomes

$$\begin{aligned}
& \overline{h^2 |U| \frac{\partial S}{\partial x}} \\
& = \left[ \bar{h}^2 \bar{U} \frac{\partial \bar{S}}{\partial x} \right] \left[ \text{Integral IX} \right] \\
& + \left[ \bar{h}^2 U_t \frac{\partial \bar{S}}{\partial x} \right] \cos\theta \left[ \text{Integral I} \right] \\
& + \left[ \bar{h}^2 U_t \frac{\partial \bar{S}}{\partial x} \right] \sin\theta \left[ \text{Integral V} \right] \\
& + \left[ \bar{h}^2 \bar{U} \frac{\partial S_t}{\partial x} \right] \left[ \text{Integral I} \right] \\
& + \left[ \bar{h}^2 U_t \frac{\partial S_t}{\partial x} \right] \cos\theta \left[ \text{Integral II} \right] \\
& + \left[ \bar{h}^2 U_t \frac{\partial S_t}{\partial x} \right] \sin\theta \left[ \text{Integral VI} \right] \\
& + \left[ 2\bar{h} h_t \bar{U} \frac{\partial \bar{S}}{\partial x} \right] \left[ \text{Integral I} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left[ 2\bar{h} \, h_t \, U_t \, \frac{\partial \bar{S}}{\partial x} \right] \cos\theta \left[ \text{Integral II} \right] \\
& + \left[ 2\bar{h} \, h_t \, U_t \, \frac{\partial \bar{S}}{\partial x} \right] \sin\theta \left[ \text{Integral VI} \right] \\
& + \left[ 2\bar{h} \, h_t \, \bar{U} \, \frac{\partial S_t}{\partial x} \right] \left[ \text{Integral II} \right] \\
& + \left[ 2\bar{h} \, h_t \, U_t \, \frac{\partial S_t}{\partial x} \right] \cos\theta \left[ \text{Integral III} \right] \\
& + \left[ 2\bar{h} \, h_t \, U_t \, \frac{\partial S_t}{\partial x} \right] \sin\theta \left[ \text{Integral VII} \right] \\
& + \left[ h_t^2 \, \bar{U} \, \frac{\partial \bar{S}}{\partial x} \right] \left[ \text{Integral II} \right] \\
& + \left[ h_t^2 \, U_t \, \frac{\partial \bar{S}}{\partial x} \right] \cos\theta \left[ \text{Integral III} \right] \\
& + \left[ h_t^2 \, U_t \, \frac{\partial \bar{S}}{\partial x} \right] \sin\theta \left[ \text{Integral VII} \right] \\
& + \left[ h_t^2 \, \bar{U} \, \frac{\partial S_t}{\partial x} \right] \left[ \text{Integral III} \right] \\
& + \left[ h_t^2 \, U_t \, \frac{\partial S_t}{\partial x} \right] \cos\theta \left[ \text{Integral IV} \right] \\
& + \left[ h_t^2 \, U_t \, \frac{\partial S_t}{\partial x} \right] \sin\theta \left[ \text{Integral VIII} \right] \tag{A1-16}
\end{aligned}$$

where

$$\begin{aligned}
\text{Integral I} &= \frac{1}{2\pi} \left[ \int_f^g \sin\phi d\phi - \int_g^j \sin\phi d\phi \right] \\
&= \frac{2}{\pi} \sqrt{1 - \frac{\bar{U}^2}{U_t^2}} \cos\theta
\end{aligned}$$

$$\begin{aligned}
 \text{Integral II} &= \frac{1}{2\pi} \left[ \int_f^g \sin^2 \phi d\phi - \int_g^j \sin^2 \phi d\phi \right] \\
 &= \frac{1}{\pi} \left[ \sin^{-1} \frac{\bar{U}}{U_t} + \frac{\bar{U}}{U_t} \sqrt{1 - \frac{\bar{U}^2}{U_t^2}} (\sin^2 \theta - \cos^2 \theta) \right]
 \end{aligned}$$

$$\begin{aligned}
 \text{Integral III} &= \frac{1}{2\pi} \left[ \int_f^g \sin^3 \phi d\phi - \int_g^j \sin^3 \phi d\phi \right] \\
 &= \frac{1}{3\pi} \left[ 2 \frac{\bar{U}^2}{U_t^2} \sqrt{1 - \frac{\bar{U}^2}{U_t^2}} \cos^3 \theta + 2 \left( 1 - \frac{\bar{U}^2}{U_t^2} \right)^{3/2} \sin^2 \theta \cos \theta \right. \\
 &\quad \left. + 4 \sqrt{1 - \frac{\bar{U}^2}{U_t^2}} \cos \theta - 4 \frac{\bar{U}^2}{U_t^2} \sqrt{1 - \frac{\bar{U}^2}{U_t^2}} \sin^2 \theta \cos \theta \right]
 \end{aligned}$$

$$\begin{aligned}
 \text{Integral IV} &= \frac{1}{2\pi} \left[ \int_f^g \sin^4 \phi d\phi - \int_g^j \sin^4 \phi d\phi \right] \\
 &= \left[ \frac{3}{4\pi} \sin^{-1} \frac{\bar{U}}{U_t} - \frac{1}{2\pi} \sin \left( 2 \sin^{-1} \frac{\bar{U}}{U_t} \right) \cos 2\theta \right. \\
 &\quad \left. + \frac{1}{16\pi} \sin \left( 4 \sin^{-1} \frac{\bar{U}}{U_t} \right) \cos 4\theta \right]
 \end{aligned}$$

$$\begin{aligned}
 \text{Integral V} &= \frac{1}{2\pi} \left[ \int_f^g \cos \phi d\phi - \int_g^j \cos \phi d\phi \right] \\
 &= \frac{2}{\pi} \sqrt{1 - \frac{\bar{U}^2}{U_t^2}} \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{Integral VI} &= \frac{1}{2\pi} \left[ \int_f^g \sin \phi \cos \phi d\phi - \int_g^j \sin \phi \cos \phi d\phi \right] \\
 &= - \frac{2}{\pi} \frac{\bar{U}}{U_t} \sqrt{1 - \frac{\bar{U}^2}{U_t^2}} \cos \theta \sin \theta
 \end{aligned}$$

$$\begin{aligned}\text{Integral VII} &= \frac{1}{2\pi} \left[ \int_f^g \sin^2 \phi \cos \phi d\phi - \int_g^j \sin^2 \phi \cos \phi d\phi \right] \\ &= \frac{1}{3\pi} \left[ 6 \frac{\bar{U}^2}{U_t^2} \sqrt{1 - \frac{\bar{U}^2}{U_t^2}} \cos^2 \theta \sin \theta + 2 \left( 1 - \frac{\bar{U}^2}{U_t^2} \right)^{3/2} \sin^3 \theta \right]\end{aligned}$$

$$\begin{aligned}\text{Integral VIII} &= \frac{1}{2\pi} \left[ \int_f^g \sin^3 \phi \cos \phi d\phi - \int_g^j \sin^3 \phi \cos \phi d\phi \right] \\ &= -\frac{2}{\pi} \frac{\bar{U}^3}{U_t^3} \sqrt{1 - \frac{\bar{U}^2}{U_t^2}} \cos^3 \theta \sin^3 \theta - \frac{2}{\pi} \frac{\bar{U}}{U_t} \left( 1 - \frac{\bar{U}^2}{U_t^2} \right)^{3/2} \sin^3 \theta \cos \theta\end{aligned}$$

$$\begin{aligned}\text{Integral IX} &= \frac{1}{2\pi} \left[ \int_f^g d\phi - \int_g^j d\phi \right] \\ &= \frac{2}{\pi} \sin^{-1} \frac{\bar{U}}{U_t}\end{aligned}$$

Therefore, by using approximations (A1-8a) and neglecting the second and higher order terms, equation (A1-16) reduces to

$$\begin{aligned}&\overline{h^2 |U| \frac{\partial \bar{S}}{\partial x}} \\ &= \frac{2}{\pi} \left( \sin^{-1} \frac{\bar{U}}{U_t} \right) \left[ \bar{h}^2 \bar{U} \frac{\partial \bar{S}}{\partial x} \right] \\ &\quad + \frac{2}{\pi} \sqrt{1 - \frac{\bar{U}^2}{U_t^2}} \left[ \bar{h}^2 U_t \frac{\partial \bar{S}}{\partial x} \right] \\ &= \frac{2}{\pi} \left[ \left( \sin^{-1} \frac{\bar{U}}{U_t} \right) \bar{U} + \sqrt{1 - \frac{\bar{U}^2}{U_t^2}} U_t \right] \bar{h}^2 \frac{\partial \bar{S}}{\partial x} \\ &= \bar{h}^2 \overline{|U|} \frac{\partial \bar{S}}{\partial x}\end{aligned}$$

$$\begin{aligned}
 \text{since } \overline{|U|} &= \overline{|\bar{U} + U_t \sin(\frac{2\pi}{T} t + \theta)|} \\
 &= \frac{2}{\pi} \left[ \left( \sin^{-1} \frac{\bar{U}}{U_t} \right) \bar{U} + \sqrt{1 - \frac{\bar{U}^2}{U_t^2}} U_t \right]
 \end{aligned}$$

$$\begin{aligned}
 \text{Thus } -kBh^2 \overline{|U|} \frac{\partial \bar{S}}{\partial x} &= -kB\bar{h}^2 \overline{|U|} \frac{\partial \bar{S}}{\partial x} \\
 &= -k\bar{A} \bar{h} \overline{|U|} \frac{\partial \bar{S}}{\partial x} \\
 &= -\bar{A} \bar{E} \frac{\partial \bar{S}}{\partial x}
 \end{aligned}$$

$$\text{and } \bar{E} = k \bar{h} \overline{|U|}$$

## APPENDIX A2

### FORMULATION OF MODELS WITH SLACK TIDE

#### APPROXIMATIONS

The instantaneous salt concentration can be represented in terms of  $S_h$ , the salt concentration at high water slack, by the expression

$$S = S_h - S_h' \quad (A2-1)$$

where  $S_h'$  is the deviation of salinity from that of high water slack and  $\overline{S_h'} \neq 0$ . Also, let the cross-sectional averages of the velocity  $U$  and area  $A$  be defined as in Chapter II, where

$$\begin{aligned} U &= \bar{U} + U' \\ A &= \bar{A} + A' \end{aligned} \quad (A2-2)$$

Now, by substituting equations (A2-1, 2) into the mass balance equation (II-1), we have

$$\begin{aligned} \frac{\partial}{\partial t} [(\bar{A} + A') (S_h - S_h')] + \frac{\partial}{\partial x} [(\bar{A} + A') (\bar{U} + U') (S_h - S_h')] \\ = \frac{\partial}{\partial x} [(\bar{A} + A') E \frac{\partial}{\partial x} (S_h - S_h')] \end{aligned} \quad (A2-3)$$

Next expand equation (A2-3) and average it over a tidal cycle. We then find that



$$\begin{aligned}
& \frac{\partial}{\partial t} \left[ \overline{\bar{A}S_h} - \overline{\bar{A}S_{h'}} + \overline{A'S_h} - \overline{A'S_{h'}} \right] \\
& + \frac{\partial}{\partial x} \left[ \overline{\bar{A}\bar{U}S_h} - \overline{\bar{A}\bar{U}S_{h'}} + \overline{\bar{A}U'S_h} - \overline{\bar{A}U'S_{h'}} + \overline{A'\bar{U}S_h} - \overline{A'\bar{U}S_{h'}} \right. \\
& \quad \left. + \overline{A'U'S_h} - \overline{A'U'S_{h'}} \right] \\
& = \frac{\partial}{\partial x} \left[ \overline{\bar{A}\bar{E} \frac{\partial S_h}{\partial x}} - \overline{\bar{A}\bar{E} \frac{\partial S_{h'}}{\partial x}} + \overline{A'E \frac{\partial S_h}{\partial x}} - \overline{A'E \frac{\partial S_{h'}}{\partial x}} \right] \quad (A2-4)
\end{aligned}$$

The tidal averages of the cross-sectional area  $\bar{A}$  and the velocity  $\bar{U}$  are assumed quasi-stationary as in the case where  $S = \bar{S} + S'$ . The following assumptions are also made:

$$S_{h'} \ll S_h$$

$$\text{and } A' \ll \bar{A}.$$

Thus, neglecting the second and higher order terms of  $S_{h'}$  and  $A'$ , equation (A2-4) reduces to

$$\begin{aligned}
& \frac{\partial}{\partial t} \left[ \overline{\bar{A}S_h} - \overline{\bar{A}S_{h'}} \right] + \frac{\partial}{\partial x} \left[ \overline{\bar{A}\bar{U}S_h} - \overline{\bar{A}\bar{U}S_{h'}} - \overline{\bar{A}U'S_{h'}} + \overline{A'U'S_h} \right] \\
& = \frac{\partial}{\partial x} \left[ \overline{\bar{A}\bar{E} \frac{\partial S_h}{\partial x}} - \overline{\bar{A}\bar{E} \frac{\partial S_{h'}}{\partial x}} \right] \quad (A2-5)
\end{aligned}$$

A similar calculation, as that of Appendix A1, will show that

$$\overline{\bar{A}\bar{E} \frac{\partial S_h}{\partial x}} = \bar{A}\bar{E} \frac{\partial S_h}{\partial x}$$

to the first order, where  $\bar{E} = k\bar{h} \overline{|U|}$

The continuity equation, averaged over a tidal cycle, yields as before

$$\frac{\partial}{\partial t} \bar{A} + \frac{\partial}{\partial x} [\bar{A}\bar{U} + \overline{A'U'}] = 0$$

$$\text{or } \frac{\partial}{\partial t} \bar{A} + \frac{\partial}{\partial x} [\bar{A}U_f] = 0 \quad (\text{A2-6})$$

$$\text{where } U_f = \bar{U} + \frac{\overline{A'U'}}{\bar{A}} \quad (\text{A2-7})$$

So, by combining equations (A2-5) and (A2-6), the mass balance equation may be simplified to

$$\begin{aligned} & \bar{A} \frac{\partial S_h}{\partial t} - \frac{\partial}{\partial t} (\bar{A}\overline{S_h'}) + \bar{A}U_f \frac{\partial S_h}{\partial x} + \frac{\partial}{\partial x} [-\bar{A}\overline{US_h'}] \\ &= \frac{\partial}{\partial x} \left[ \bar{A}\bar{E} \frac{\partial S_h}{\partial x} - \bar{A}\bar{E} \frac{\partial \overline{S_h'}}{\partial x} \right] + \frac{\partial}{\partial x} [\bar{A}\overline{U'S_h'}] \end{aligned} \quad (\text{A2-8})$$

Next, divide equation (A2-8) by  $\bar{A}$  and assume that

$$\overline{U'S_h'} = E_{th} \frac{\partial S_h}{\partial x} \quad (\text{A2-9})$$

This gives

$$\begin{aligned} & \frac{\partial S_h}{\partial t} + U_f \frac{\partial S_h}{\partial x} - \frac{1}{\bar{A}} \left[ \frac{\partial}{\partial t} (\bar{A}\overline{S_h'}) + \frac{\partial}{\partial x} (\bar{A}\overline{US_h'}) \right] \\ &= \frac{1}{\bar{A}} \frac{\partial}{\partial x} \left[ \bar{A} (\bar{E} + E_{th}) \frac{\partial S_h}{\partial x} \right] - \frac{\partial}{\partial x} (\bar{A}\bar{E} \frac{\partial \overline{S_h'}}{\partial x}) \end{aligned} \quad (\text{A2-10})$$

To arrive at an equation for the salinity  $S_h$ , it is assumed that both  $S_h$  and  $\overline{S_h'}$  satisfy the convection-diffusion equation separately, i.e.

$$\frac{\partial}{\partial t} (\bar{A} \bar{S}_h') + \frac{\partial}{\partial x} (\bar{A} \bar{U} \bar{S}_h') = \frac{\partial}{\partial x} (\bar{A} \bar{E} \frac{\partial \bar{S}_h'}{\partial x}) \quad (\text{A2-11})$$

and

$$\frac{\partial}{\partial t} S_h + U_f \frac{\partial S_h}{\partial x} = \frac{1}{\bar{A}} \frac{\partial}{\partial x} (\bar{A} E_h \frac{\partial S_h}{\partial x}) \quad (\text{A2-12})$$

$$\text{where } E_h = \bar{E} + E_{th} \quad (\text{A2-13})$$

$$\text{and } E_{th} = \bar{U}' S_h' \left[ \frac{\partial S_h}{\partial x} \right]^{-1} \quad (\text{A2-14})$$

Similarly, the instantaneous salt concentration can be represented in terms of  $S_L$ , the salt concentration at low water slack by the expression

$$S = S_L + S_L' \quad (\text{A2-15})$$

where  $S_L'$  is the deviation of salinity from that of low water slack and  $\bar{S}_L' \neq 0$ . When equation (A2-15) is substituted into equation (II-1), the mass balance equation (II-1) becomes

$$\begin{aligned} \frac{\partial}{\partial t} [(\bar{A} + A') (S_L + S_L')] + \frac{\partial}{\partial x} [(\bar{A} + A') (\bar{U} + U') (S_L + S_L')] \\ = \frac{\partial}{\partial x} [(\bar{A} + A') E \frac{\partial}{\partial x} (S_L + S_L')] \end{aligned} \quad (\text{A2-16})$$

However, when equation (A2-16) is expanded and averaged over a tidal cycle, we have

$$\begin{aligned} \frac{\partial}{\partial t} [\bar{A} \bar{S}_L + \bar{A} \bar{S}_L' + \bar{A}' \bar{S}_L + \bar{A}' \bar{S}_L'] \\ + \frac{\partial}{\partial x} [\bar{A} \bar{U} \bar{S}_L + \bar{A} \bar{U} \bar{S}_L' + \bar{A}' \bar{U}' \bar{S}_L + \bar{A}' \bar{U}' \bar{S}_L' + \bar{A}' \bar{U} \bar{S}_L + \bar{A}' \bar{U} \bar{S}_L' \\ + \bar{A}' \bar{U}' \bar{S}_L + \bar{A}' \bar{U}' \bar{S}_L'] \\ = \frac{\partial}{\partial x} \left[ \bar{A} \bar{E} \frac{\partial \bar{S}_L}{\partial x} + \bar{A} \bar{E} \frac{\partial \bar{S}_L'}{\partial x} + \bar{A}' \bar{E} \frac{\partial \bar{S}_L}{\partial x} + \bar{A}' \bar{E} \frac{\partial \bar{S}_L'}{\partial x} \right] \end{aligned} \quad (\text{A2-17})$$

The tidal averages of the cross-sectional area  $\bar{A}$  and the velocity  $\bar{U}$  are assumed quasi-stationary during a tidal period and  $S_L' \ll S_L$  and  $A' \ll \bar{A}$ . Thus, neglecting the second and higher order terms of  $S_L'$  and  $A'$ , equation (A2-17) reduces to

$$\begin{aligned} & \frac{\partial}{\partial t} [\bar{A} S_L + \bar{A} \overline{S_L'}] + \frac{\partial}{\partial x} [\bar{A} \bar{U} S_L + \bar{A} \bar{U} \overline{S_L'} + \bar{A} \overline{U' S_L'} + \bar{A}' \overline{U' S_L}] \\ &= \frac{\partial}{\partial x} \left[ \bar{A} \bar{E} \frac{\partial S_L}{\partial x} + \bar{A} \bar{E} \frac{\partial \overline{S_L'}}{\partial x} \right] \end{aligned} \quad (\text{A2-18})$$

Furthermore, by using the continuity equation (A2-6), equation (A2-18) reduces to

$$\begin{aligned} & \bar{A} \frac{\partial S_L}{\partial t} + \bar{A} U_f \frac{\partial S_L}{\partial x} + \frac{\partial}{\partial t} (\bar{A} \overline{S_L'}) + \frac{\partial}{\partial x} (\bar{A} \bar{U} \overline{S_L'}) \\ &= \frac{\partial}{\partial x} \left[ \bar{A} \bar{E} \frac{\partial S_L}{\partial x} - \bar{A} \overline{U' S_L'} \right] + \frac{\partial}{\partial x} \left[ \bar{A} \bar{E} \frac{\partial \overline{S_L'}}{\partial x} \right] \end{aligned} \quad (\text{A2-19})$$

Finally, both  $S_L$  and  $\overline{S_L'}$  are assumed to satisfy the convection-diffusion equation separately and

$$- \overline{U' S_L'} = E_{tL} \frac{\partial S_L}{\partial x} \quad (\text{A2-20})$$

Thus, equation (A2-19) becomes

$$\frac{\partial}{\partial t} (\bar{A} \overline{S_L'}) + \frac{\partial}{\partial x} (\bar{A} \bar{U} \overline{S_L'}) = \frac{\partial}{\partial x} \left( \bar{A} \bar{E} \frac{\partial \overline{S_L'}}{\partial x} \right) \quad (\text{A2-21})$$

and

$$\frac{\partial S_L}{\partial t} + U_f \frac{\partial S_L}{\partial x} = \frac{1}{\bar{A}} \frac{\partial}{\partial x} \left[ \bar{A} E_L \frac{\partial S_L}{\partial x} \right] \quad (\text{A2-22})$$

$$\text{where} \quad E_L = \bar{E} + E_{tL} \quad (\text{A2-23})$$

$$\text{and} \quad E_{tL} = - \overline{U' S_L'} \left[ \frac{\partial S_L}{\partial x} \right]^{-1} \quad (\text{A2-24})$$

# APPENDIX A3

$$\text{Let } U = \bar{U} + U'$$

$$= \bar{U} + U_t \sin \left( \frac{2\pi}{T} t + \theta \right) \quad (\text{A3-1})$$

and let the salinity be represented as follows:

$$S = \bar{S} + S'$$

$$= \bar{S} + S_t \sin \frac{2\pi}{T} t \quad (\text{A3-2})$$

for the tidal average model,

$$\begin{aligned} S &= (\bar{S} + S_t) - (S_t - S_t \sin \frac{2\pi}{T} t) \\ &= S_h - S_h' \end{aligned} \quad (\text{A3-3})$$

for the high water slack approximation, and

$$\begin{aligned} S &= (\bar{S} - S_t) + (S_t + S_t \sin \frac{2\pi}{T} t) \\ &= S_L + S_L' \end{aligned} \quad (\text{A3-4})$$

for the low water slack approximation.

The quantity  $\overline{U'S_*'}$  ( $S_*'$  is  $S'$  or  $S_h'$  or  $S_L'$ ) is evaluated for each case and the calculations are presented below.

Average over a tidal cycle

$$\begin{aligned}
\overline{U'S} &= \frac{1}{T} \int_{-T/2}^{T/2} U_t \sin\left(\frac{2\pi}{T}t + \theta\right) S_t \sin\frac{2\pi}{T}t \, dt \\
&= \frac{1}{T} \int_{-T/2}^{T/2} U_t S_t \cos\theta \sin^2 \frac{2\pi}{T}t \, dt \\
&\quad + \frac{1}{T} \int_{-T/2}^{T/2} U_t S_t \sin\theta \sin\frac{2\pi}{T}t \cos\frac{2\pi}{T}t \, dt \\
&= \frac{U_t S_t}{2} \cos\theta
\end{aligned} \tag{A3-5}$$

High Water Slack Approximation

$$\begin{aligned}
\overline{U'S_h} &= \frac{1}{T} \int_{-T/2}^{T/2} U_t \sin\left(\frac{2\pi}{T}t + \theta\right) \left[S_t - S_t \sin\frac{2\pi}{T}t\right] \, dt \\
&= \frac{1}{T} \int_{-T/2}^{T/2} U_t S_t \sin\left(\frac{2\pi}{T}t + \theta\right) \, dt \\
&\quad - \frac{1}{T} \int_{-T/2}^{T/2} U_t S_t \sin\frac{2\pi}{T}t \sin\left(\frac{2\pi}{T}t + \theta\right) \, dt \\
&= - \frac{U_t S_t}{2} \cos\theta
\end{aligned} \tag{A3-6}$$

Low Water Slack Approximation

$$\begin{aligned}
\overline{U'S_L'} &= \frac{1}{T} \int_{-T/2}^{T/2} U_t \sin \left( \frac{2\pi}{T}t + \theta \right) \left[ S_t + S_t \sin \frac{2\pi}{T}t \right] dt \\
&= \frac{1}{T} \int_{-T/2}^{T/2} U_t S_t \sin \frac{2\pi}{T}t \sin \left( \frac{2\pi}{T}t + \theta \right) dt \\
&\quad + \frac{1}{T} \int_{-T/2}^{T/2} U_t S_t \sin \left( \frac{2\pi}{T}t + \theta \right) dt \\
&= \frac{U_t S_t}{2} \cos \theta \qquad (A3-7)
\end{aligned}$$



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